Math 206B Lecture 20 Notes

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1 Generating Functions of Symmetric Functions and Cauchy-Type Identities

1.1 Relationship between e_k and h_k

We want to prove a correspondence between e_n and h_n. Define $E(t) = \sum_{n=0}^\infty e_n t^n$ and $H(t) = \sum_{n=0}^\infty h_n t^n$. So we just need to prove a relation between E and H.

Proposition 1.1. $E(t) = \prod_{i=1}^\infty (1 + x_i t)$.

Proof. This follows from $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$.[1] $\square$

Proposition 1.2. $H(t) = \prod_{i=1}^\infty \frac{1}{1 - x_i t}$.

Proof. This follows from $h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}$. $\square$

Corollary 1.1. $E(t)H(-t) = 1$.

Definition 1.1. The ω-**involution** $\omega : \Lambda \to \Lambda$ is $\omega(e_k) = h_k$ for all k.

Proposition 1.3. $\omega^2 = \mathrm{id}_\Lambda$.

Proof. This is equivalent to $\omega(h_k) = e_k$ for all k. The relation for E and H shows that $\sum_{i=0}^n (-1)^{n-i} e_i h_{n-i} = 0$ for all n. The map ω is an algebra homomorphism, so applying ω to the sum should still give 0. We can then recursively determine that $\omega(h_k) = e_k$. $\square$

Example 1.1. Say we are looking at n variable symmetric functions. Then

$$e_k(1, \dots, 1) = \binom{n}{k},$$

$$h_k(1, \dots, 1) = \binom{n+k-1}{k}.$$

[1] Igor says that this is the kind of thing he thinks is so simple, he doesn't actually know how to really prove it. He's not being facetious.

Some people call this latter quantity $\binom{-n}{k}$. The generating function of $\binom{n}{k}$ is

$$\sum_{k=0}^{\infty} \binom{n}{k} t^n = (1+t)^n,$$

while the generating function of $\binom{-n}{k}$ is

$$\sum_{k=0}^{\infty} \binom{-n}{k} t^k = (1-t)^{-n}.$$

So this is the symmetric function analogue of these classic generating function identities.

1.2 Cauchy-type identities

Consider the multivariate generating function

$$Q(\overline{x}, \overline{y}) = \prod_{i,j=1} \frac{1}{1 - x_i y_j}.$$

Proposition 1.4.

$$Q(\overline{x}, \overline{y}) = \sum_{\alpha,\beta} |\operatorname{Mat}(\alpha,\beta)| m_\alpha(x) m_\beta(y) = \sum_\lambda m_\lambda(x) h_\lambda(y),$$

where $\operatorname{Mat}(\alpha,\beta)$ *are matrices with row-sums* α *and column-sums* β*.*

Proof. For the first part, the $x^\alpha y^\beta$ term in the product is a sum of terms of products of $(x_i y_j)^{a_{i,j}}$. If we put the $a_{i,j}$ into a matrix, the matrix has row-sums α and column-sums β.

For the second part, use $h_\lambda = \sum_\alpha |\operatorname{Mat}(\alpha,\lambda)| m_\alpha(x)$. □

Proposition 1.5.

$$Q(\overline{x}, \overline{y}) = \sum_\lambda z_\lambda p_\lambda(\overline{x}) p_\lambda(\overline{y}).$$

Theorem 1.1.

$$Q(\overline{x}, \overline{y}) = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Proof. This follows from RSK. The coefficient of $x^\alpha y^\beta$ in $s_\lambda(x)s_\lambda(y)$ is $|\operatorname{SSYT}(\lambda,\alpha)| \cdot |\operatorname{SSYT}(\lambda,\beta)|$. □

When Richard Stanley wrote his textbook, he didn't believe that Cauchy proved this identity. After looking through hundreds of pages of Cauchy's works, he determined that Cauchy did not actually prove this identity. After more research, he found that Cascoux was the first person to cite the identities as theorems of Cauchy. So in fact, none of these Cauchy-type identities are due to Cauchy.

Proposition 1.6. $\{f_\alpha\}$ *is an orthonormal basis of* Λ *if*

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda f_\lambda(x) f_\lambda(y).$$

Corollary 1.2. $\{p_\lambda/\sqrt{z_\lambda}\}$ *is an orthonormal basis of* λ.